Computational Fluid Dynamics
An Introduction

Simulation in Computer Graphics
University of Freiburg

WS 04/05

Acknowledgement

This slide set is based on:

- John D. Anderson, Jr., "Computational Fluid Dynamics - The Basics with Applications," McGraw-Hill, Inc., New York, ISBN 0-07-001685-2

Motivation

D. Enright, S. Marschner, R. Fedkiw, "Animation and Rendering of Complex Water Surfaces," Siggraph 2002, ACM TOG, vol. 21, pp. 736-744, 2002.

Outline

- introduction
- pre-requisites
- governing equations
  - continuity equation
  - momentum equation
  - summary
- solution techniques
  - Lax-Wendroff
  - MacCormack
  - numerical aspects
- recent research in graphics

Introduction

- physical aspects of fluid flow are governed by three principles
  - mass is conserved
  - force = mass × acceleration (Newton's second law)
  - energy is conserved (not considered in this lecture)
- principles can be described with integral equations or partial differential equations
- in CFD, these governing equations are replaced by discretized algebraic forms
- algebraic forms provide quantities at discrete points in time and space, no closed-form analytical solution

Numerical Solution - Overview

Governing Equations → Discretization → System of Algebraic Equations → Equation Solver → Approx. Solution

Governing Equations	Discretization	System of Algebraic Equations	Equation Solver
Navier-Stokes Euler	Finite Difference	Discrete Nodal Values	Lax-Wendroff MacCormack

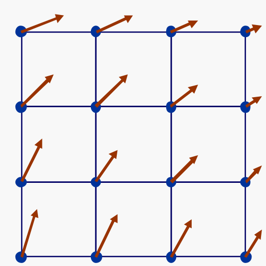
Continuous Quantities

- x, y, z - 3D coordinate system
- t - time

- *r* (x,y,z,t) - density
- v (x,y,z,t) - velocity

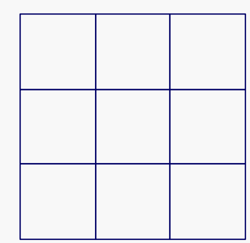
- v (x,y,z,t) = (u (x,y,z,t), v (x,y,z,t), w (x,y,z,t) $)^T$

Problem



v (*x,y,z,t*)

r (*x,y,z,t*)



v (*x,y,z,t*+Δ*t*)

r (*x,y,z,t*+Δ*t*)

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Substantial Derivative of *r*

- infinitesimally small fluid element moving with the flow

- (x_1,y_1,z_1) - position at time t_1
- (x_2,y_2,z_2) - position at time t_2

- $v_1(x_1,y_1,z_1,t_1) = (u\,(x_1,y_1,z_1,t_1),v\,(x_1,y_1,z_1,t_1),w\,(x_1,y_1,z_1,t_1))^T$
- $v_2(x_2,y_2,z_2,t_2) = (u\,(x_2,y_2,z_2,t_2),v\,(x_2,y_2,z_2,t_2),w\,(x_2,y_2,z_2,t_2))^T$

- r_1 (x_1,y_1,z_1,t_1)
- r_2 (x_2,y_2,z_2,t_2)

Substantial Derivative of *r*

- Taylor series at point 1

$$r_2 = r_1 + \left(\frac{\partial r}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial r}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial r}{\partial z}\right)_1 (z_2 - z_1) + \left(\frac{\partial r}{\partial t}\right)_1 (t_2 - t_1)$$

$\rightarrow$

$$\frac{r_2 - r_1}{t_2 - t_1} = \left(\frac{\partial r}{\partial x}\right)_1 \frac{x_2 - x_1}{t_2 - t_1} + \left(\frac{\partial r}{\partial y}\right)_1 \frac{y_2 - y_1}{t_2 - t_1} + \left(\frac{\partial r}{\partial z}\right)_1 \frac{z_2 - z_1}{t_2 - t_1} + \left(\frac{\partial r}{\partial t}\right)_1$$

$t_2 \rightarrow t_1$

$$\frac{r_2 - r_1}{t_2 - t_1} = \left(\frac{\partial r}{\partial x}\right)_1 u + \left(\frac{\partial r}{\partial y}\right)_1 v + \left(\frac{\partial r}{\partial z}\right)_1 w + \left(\frac{\partial r}{\partial t}\right)_1$$

Substantial Derivative

- substantial derivative of ρ

$$\frac{Dr}{Dt} \equiv u\frac{\partial r}{\partial x} + v\frac{\partial r}{\partial y} + w\frac{\partial r}{\partial z} + \frac{\partial r}{\partial t}$$

- substantial derivative = local derivative + convective derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) \qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$$

- local derivative - time rate of change at a fixed location
- convective derivative - time rate of change due to fluid flow
- subst. derivative = total derivative with respect to time

Divergence of *v*

- divergence of velocity v = time rate of change of the volume δV of a moving fluid element per unit volume

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{1}{dV} \frac{D(dV)}{Dt}$$

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Continuity Equation

- mass is conserved
- infinitesimally small fluid element moving with the flow
- fixed mass δm, variable volume δV, variable density ρ

  $dm = r \cdot dV$
- time rate of change of the mass of the moving fluid element is zero

$$\frac{Ddm}{Dt} = \frac{D(r \cdot dV)}{Dt} = dV \frac{Dr}{Dt} + r \frac{DdV}{Dt} = 0$$

$$\frac{Dr}{Dt} + r \frac{1}{dV} \frac{DdV}{Dt} = 0 \quad \rightarrow \quad \boxed{\frac{Dr}{Dt} + r \nabla \cdot \mathbf{v} = 0}$$

Continuity Equation

$$\frac{Dr}{Dt} + r \nabla \cdot \mathbf{v} = 0$$

substantial derivative - time rate of change of density of a moving fluid element

divergence of velocity - time rate of change of volume of a moving fluid element per volume

Continuity Equation

- non-conservation form (considers moving element)

$$\boxed{\frac{D r}{Dt} + r \nabla \cdot \mathbf{v} = 0}$$

- manipulation

$$\frac{D r}{Dt} + r \nabla \cdot \mathbf{v} = \frac{\partial r}{\partial t} + \mathbf{v} \cdot \nabla r + r \nabla \cdot \mathbf{v} = \frac{\partial r}{\partial t} + \nabla \cdot (r \mathbf{v})$$

- conservation form (considers element at fixed location)

$$\boxed{\frac{\partial r}{\partial t} + \nabla \cdot (r \mathbf{v}) = 0}$$

Continuity Equation

- motivation for conservation form
  - infinitesimally small element at a fixed location
  - mass flux through element
  - difference of mass inflow and outflow = net mass flow
  - net mass flow = time rate of mass increase

$$\frac{\partial r}{\partial t} + \nabla \cdot (r \mathbf{v}) = 0$$

time rate of mass increase per volume

net mass flow per volume

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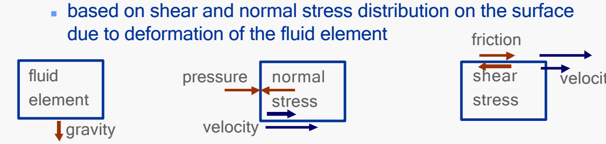
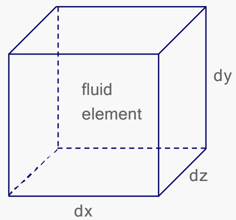
Momentum Equation

- consider a moving fluid element
- physical principle: F = m·a (Newton's second law)
- two sources of forces
  - body forces
    - gravity
  - surface forces
    - based on pressure distribution on the surface
    - based on shear and normal stress distribution on the surface due to deformation of the fluid element



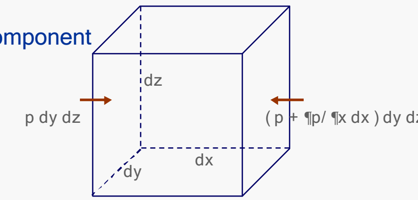
Body Force

- f = body force per unit mass
- gravity: f = δm·g / δm = g
  - ® body force on fluid element
    ρ f (dx dy dz)



Pressure Force

- consider the x component
- pressure force acts orthogonal to surface into the fluid element

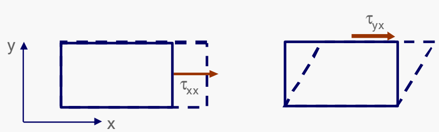


- net pressure force in x direction

$$\left[p-\left(p+\frac{\partial p}{\partial x}dx\right)\right]dy\ dz=-\frac{\partial p}{\partial x}dx\ dy\ dz$$

Stress

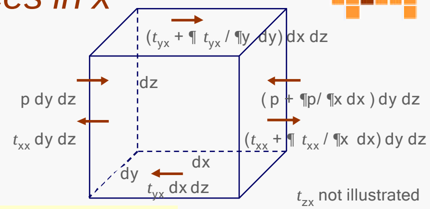
- normal stress - related to the time rate of change of volume
- shear stress - related to the time rate of change of the shearing deformation
- τ_{jk} - stress in k direction applied to a surface perpendicular to the j axis
- τ_{xx} - normal stress in x direction
- τ_{zx} , τ_{yx} - shear stresses in x directions



Stress

- normal stress is related to pressure orthogonal to surface
- shear stress is related to friction parallel to surface
- friction and pressure are related to the velocity gradient
  - friction (shear stress) models viscosity (viscous flow)
  - in contrast to inviscid flow, where friction is not considered

Surface Forces in x



t_{zx} not illustrated

$$F_x = \left[p - \left(p + \frac{\partial p}{\partial x}dx\right)\right]dy\,dz + \left[\left(t_{xx} + \frac{\partial t_{xx}}{\partial x}dx\right) - t_{xx}\right]dy\,dz$$
$$+\left[\left(t_{yx} + \frac{\partial t_{yx}}{\partial y}dy\right) - t_{yx}\right]dx\,dz + \left[\left(t_{zx} + \frac{\partial t_{zx}}{\partial z}dz\right) - t_{zx}\right]dy\,dz$$

$$F_x = \left(-\frac{\partial p}{\partial x} + \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{yx}}{\partial y} + \frac{\partial t_{zx}}{\partial z}\right)dx\,dy\,dz$$

Momentum Equation in x

$$m\,a_x = F_x$$

$$r\,dx\,dy\,dz\,\frac{Du}{Dt} = \left(-\frac{\partial p}{\partial x} + \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{yx}}{\partial y} + \frac{\partial t_{zx}}{\partial z}\right)dx\,dy\,dz + r\,f_x\,dx\,dy\,dz$$

mass	acceleration	surface force	body force
	time rate of change of velocity of a moving fluid element	pressure normal stress shear stress	gravity

$$r\,\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{yx}}{\partial y} + \frac{\partial t_{zx}}{\partial z} + r\,f_x$$

Momentum Equation

- viscous flow, non-conversation form
- Navier-Stokes equations

$$\boxed{r\,\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{yx}}{\partial y} + \frac{\partial t_{zx}}{\partial z} + r\,f_x}$$

$$\boxed{r\,\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{zy}}{\partial z} + r\,f_y}$$

$$\boxed{r\,\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y} + \frac{\partial t_{zz}}{\partial z} + r\,f_z}$$

Momentum Equation in x

- obtaining the conservation form of Navier-Stokes

$$r\,\frac{Du}{Dt} = r\,\frac{\partial u}{\partial t} + r\,\mathbf{v}\cdot\nabla u$$

$$\frac{\partial(ru)}{\partial t} = r\,\frac{\partial u}{\partial t} + u\,\frac{\partial r}{\partial t} \longrightarrow r\,\frac{\partial u}{\partial t} = u\,\frac{\partial r}{\partial t} - \frac{\partial(ru)}{\partial t}$$

$$\nabla\cdot(r\,u\,\mathbf{v}) = u\nabla\cdot(r\,\mathbf{v}) + (r\,\mathbf{v})\nabla u \longrightarrow (r\,\mathbf{v})\nabla u = \nabla\cdot(r\,u\,\mathbf{v}) - u\nabla\cdot(r\,\mathbf{v})$$

$$r\,\frac{Du}{Dt} = \frac{\partial(ru)}{\partial t} - u\underbrace{\left[\frac{\partial r}{\partial t} + \nabla\cdot(r\,\mathbf{v})\right]}_{\text{continuity equation}} + \nabla\cdot(r\,u\,\mathbf{v}) \longrightarrow r\,\frac{Du}{Dt} = \frac{\partial(ru)}{\partial t} + \nabla\cdot(r\,u\,\mathbf{v})$$

Momentum Equation

- Navier-Stokes equations in conservation form

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{v}) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{v}) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$$

$$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \mathbf{v}) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$$

Newtonian Fluids

- Newton: shear stress in a fluid is proportional to velocity gradients
- most fluids can be assumed to be newtonian, but blood is a popular non-newtonian fluid
- Stokes:

$$\tau_{xx} = \lambda(\nabla \cdot \mathbf{v}) + 2\mu \frac{\partial u}{\partial x} \quad \tau_{yy} = \lambda(\nabla \cdot \mathbf{v}) + 2\mu \frac{\partial v}{\partial y} \quad \tau_{zz} = \lambda(\nabla \cdot \mathbf{v}) + 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \quad \tau_{xz} = \tau_{zx} = \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \quad \tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]$$

- μ is the molecular viscosity coefficient
- λ is the second viscosity coefficient with $\lambda = -2/3 \cdot \mu$

Perfect Gas

- intermolecular forces are negligible
- R - specific gas constant
- T - temperature
- termal equation of state $p = \rho R T$

Momentum Equation in x

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{v}) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

- now,
  p can be expressed using density and
  τ can be expressed using velocity gradients,
  f_x is user-defined (e. g. gravity)

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Governing Equations – Viscous Compressible Flow

- thus, four equations and four unknowns ρ, u, v, w

$$\frac{\partial r}{\partial t} + \nabla \cdot (r\,\mathbf{v}) = 0$$

continuity equation

$$\frac{\partial (r\,u)}{\partial t} + \nabla \cdot (r\,u\,\mathbf{v}) = -\frac{\partial p}{\partial x} + \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{yx}}{\partial y} + \frac{\partial t_{zx}}{\partial z} + r\,f_x$$

$$\frac{\partial (r\,v)}{\partial t} + \nabla \cdot (r\,v\,\mathbf{v}) = -\frac{\partial p}{\partial y} + \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{zy}}{\partial z} + r\,f_y$$

$$\frac{\partial (r\,w)}{\partial t} + \nabla \cdot (r\,w\,\mathbf{v}) = -\frac{\partial p}{\partial z} + \frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y} + \frac{\partial t_{zz}}{\partial z} + r\,f_z$$

momentum equation

Governing Equations – Inviscid Compressible Flow

- Euler equations

$$\frac{\partial r}{\partial t} + \nabla \cdot (r\,\mathbf{v}) = 0$$

continuity equation

$$\frac{\partial (r\,u)}{\partial t} + \nabla \cdot (r\,u\,\mathbf{v}) = -\frac{\partial p}{\partial x} + r\,f_x$$

$$\frac{\partial (r\,v)}{\partial t} + \nabla \cdot (r\,v\,\mathbf{v}) = -\frac{\partial p}{\partial y} + r\,f_y$$

$$\frac{\partial (r\,w)}{\partial t} + \nabla \cdot (r\,w\,\mathbf{v}) = -\frac{\partial p}{\partial z} + r\,f_z$$

momentum equation

Comments on the Governing Equations

- coupled system of nonlinear partial differential equations
- normal and shear stress terms are functions of the velocity gradients
- pressure is a function of the density
- only momentum equations are Navier-Stokes equations, however the name is commonly used for the full set of governing equations (plus energy equation)

Governing Equations in Conservation Form

- can be obtained directly from a control volume fixed in space
- consider flux of mass, momentum (and energy) into and out of the volume
- have a divergence term which involves the flux of mass (ρv), momentum in x, y, z (ρu v, ρv v, ρw v)
- can be expressed in a generic form

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}+\frac{\partial \mathbf{H}}{\partial z}=\mathbf{J}$$

Generic Form

- viscous flow

$$\mathbf{U}=\begin{pmatrix} r \\ r\,u \\ r\,v \\ r\,w \end{pmatrix}\quad \mathbf{F}=\begin{pmatrix} r\,u \\ r\,uu+p-t_{xx} \\ r\,vu-t_{xy} \\ r\,wu-t_{xz} \end{pmatrix}\quad \mathbf{G}=\begin{pmatrix} ?v \\ ?u\,v-t_{yx} \\ ?v\,v+p-t_{yy} \\ ?\,wv-t_{yz} \end{pmatrix}\quad \mathbf{H}=\begin{pmatrix} ?w \\ ?u\,w-t_{zx} \\ ?v\,w+p-t_{zy} \\ ?\,ww+p-t_{zz} \end{pmatrix}\quad \mathbf{J}=\begin{pmatrix} 0 \\ ?\,f_x \\ ?\,f_y \\ ?\,f_z \end{pmatrix}$$

- inviscid flow

$$\mathbf{U}=\begin{pmatrix} r \\ r\,u \\ r\,v \\ r\,w \end{pmatrix}\quad \mathbf{F}=\begin{pmatrix} r\,u \\ r\,uu+p \\ r\,vu \\ r\,wu \end{pmatrix}\quad \mathbf{G}=\begin{pmatrix} ?v \\ ?u\,v \\ ?v\,v+p \\ ?w\,v \end{pmatrix}\quad \mathbf{H}=\begin{pmatrix} ?w \\ ?u\,w \\ ?v\,w+p \\ ?w\,w+p \end{pmatrix}\quad \mathbf{J}=\begin{pmatrix} 0 \\ ?\,f_x \\ ?\,f_y \\ ?\,f_z \end{pmatrix}$$

Generic Form

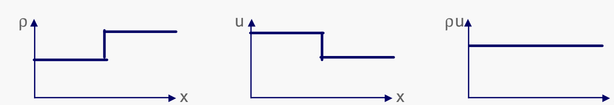
- U - solution vector (to be solved for)
- F, G, H - flux vectors
- J - source vector (external forces, energy changes)

$$\frac{\partial \mathbf{U}}{\partial t}=\mathbf{J}-\frac{\partial \mathbf{F}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}+\frac{\partial \mathbf{H}}{\partial z}$$

- dependent flow field variables can be solved progressively in steps of time (time-marching)
- spatial derivatives are know from previous time steps
- numbers can be obtained for density ρ and flux variables ρu, ρv, ρw

Motivation for Different Forms of the Governing Equations

- in many real-world problems, discontinuous changes in the flow-field variables ρ, v occur (shocks, shock waves)
- problem: differential form of the governing equations assumes differentiable (continuous) flow properties
- simple 1-D shock wave: $\rho_1\ u_1 = \rho_2\ u_2$



- conservation form solves for ρu, sees no discontinuity
- positive effect on numerical accuracy and stability

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Lax-Wendroff Technique

- flow-field variables ρ, u, v, w are known at each discrete spatial position (x, y, z) at time t
- Lax-Wendroff computes all information at time t +Δt
- second-order accuracy in space and time

- explicit, finite-difference method
- particularly suited to marching solutions
- marching suited to hyperbolic, parabolic PDEs
- unsteady (time-dependent), compressible flow is governed by hyperbolic PDEs

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Lax-Wendroff Example

- 2D, inviscid flow, no body force, in non-conservation form

$$\frac{\partial r}{\partial t} = -\left(r\frac{\partial u}{\partial x} + u\frac{\partial r}{\partial x} + r\frac{\partial v}{\partial y} + v\frac{\partial r}{\partial y} \right)$$

$$\frac{\partial u}{\partial t} = -\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{RT}{r}\frac{\partial r}{\partial x} \right)$$

$$\frac{\partial v}{\partial t} = -\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{RT}{r}\frac{\partial r}{\partial y} \right)$$

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Taylor Series Expansions

- $\rho^t_{i,j}$ - density at position (i, j) and time t
- $\rho^t_{i,j}$, $u^t_{i,j}$, $v^t_{i,j}$ are known

$$r^{t+\Delta t}_{i,j} = r^t_{i,j} + \left(\frac{\partial r}{\partial t}\right)^t_{i,j} \Delta t + \left(\frac{\partial^2 r}{\partial t^2}\right)^t_{i,j} \frac{(\Delta t)^2}{2} + \ldots$$

$$u^{t+\Delta t}_{i,j} = u^t_{i,j} + \left(\frac{\partial u}{\partial t}\right)^t_{i,j} \Delta t + \left(\frac{\partial^2 u}{\partial t^2}\right)^t_{i,j} \frac{(\Delta t)^2}{2} + \ldots$$

$$v^{t+\Delta t}_{i,j} = v^t_{i,j} + \left(\frac{\partial v}{\partial t}\right)^t_{i,j} \Delta t + \left(\frac{\partial^2 v}{\partial t^2}\right)^t_{i,j} \frac{(\Delta t)^2}{2} + \ldots$$

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Taylor Expansion - Density

$$r_{i,j}^{t+\Delta t} = r_{i,j}^{t} + \left(\frac{\partial r}{\partial t}\right)_{i,j}^{t} \Delta t + \left(\frac{\partial^2 r}{\partial t^2}\right)_{i,j}^{t} \frac{(\Delta t)^2}{2} + \ldots$$

- $\partial\rho/\partial t$ can be replaced by spatial derivatives given in the governing equations

$$\frac{\partial r}{\partial t} = -\left(r \frac{\partial u}{\partial x} + u \frac{\partial r}{\partial x} + r \frac{\partial v}{\partial y} + v \frac{\partial r}{\partial y} \right)$$

- using second-order central differences

$$\left(\frac{\partial r}{\partial t}\right)_{i,j}^{t} = -\left(r_{i,j}^{t} \frac{u_{i+1,j}^{t} - u_{i-1,j}^{t}}{2\Delta x} + u_{i,j}^{t} \frac{r_{i+1,j}^{t} - r_{i-1,j}^{t}}{2\Delta x} + r_{i,j}^{t} \frac{v_{i+1,j}^{t} - v_{i-1,j}^{t}}{2\Delta y} + v_{i,j}^{t} \frac{r_{i+1,j}^{t} - r_{i-1,j}^{t}}{2\Delta y} \right)$$

Taylor Expansion - Density

- $\partial^2\rho/\partial t^2$ can be obtained by differentiating the governing equation with respect to time

$$\frac{\partial^2 r}{\partial t^2} = -\left(r \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial u}{\partial x}\frac{\partial r}{\partial t} + u \frac{\partial^2 r}{\partial x \partial t} + \frac{\partial r}{\partial x}\frac{\partial u}{\partial t} + r \frac{\partial^2 v}{\partial y \partial t} + \frac{\partial v}{\partial y}\frac{\partial r}{\partial t} + v \frac{\partial^2 r}{\partial y \partial t} + \frac{\partial r}{\partial y}\frac{\partial v}{\partial t} \right)$$

- mixed second derivatives can be obtained by differentiating the governing equations with respect to a spatial variable, e. g.

$$\frac{\partial^2 u}{\partial x \partial t} = -\left(u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{RT}{r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{RT}{r}\frac{\partial^2 r}{\partial x^2} \right)$$

- brackets missing on pp. 219, 220 in Anderson's book!

Taylor Expansion - Density

- $\rightarrow$ $\partial^2\rho/\partial t^2$ can be computed using central differences for spatial derivatives
- higher-order terms are required, e. g.

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}^{t} = \frac{u_{i+1,j}^{t} - 2u_{i,j}^{t} + u_{i-1,j}^{t}}{(\Delta x)^2}$$

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j}^{t} = \frac{u_{i+1,j+1}^{t} + u_{i-1,j-1}^{t} - u_{i-1,j+1}^{t} - u_{i+1,j-1}^{t}}{4\Delta x \Delta y}$$

- ... u, v are computed in the same way

Outline

- introduction
- pre-requisites
- governing equations
  - continuity equation
  - momentum equation
  - summary
- solution techniques
  - Lax-Wendroff
  - MacCormack
  - numerical aspects
- recent research in graphics

MacCormack Technique

- same characteristics like Lax-Wendroff
- second-order accuracy in space and time
- requires only first time derivative
- predictor - corrector
- illustrated for density

$$r_{i,j}^{t+\Delta t} = r_{i,j}^{t} + \left(\frac{\partial r}{\partial t}\right)_{i,j}^{av} \Delta t$$

MacCormack Technique

- predictor step for density

$$\left(\frac{\partial r}{\partial t}\right)_{i,j}^{t} = -\left(r_{i,j}^{t} \frac{u_{i+1,j}^{t} - u_{i-1,j}^{t}}{2\Delta x} + u_{i,j}^{t} \frac{r_{i+1,j}^{t} - r_{i-1,j}^{t}}{2\Delta x} + r_{i,j}^{t} \frac{v_{i+1,j}^{t} - v_{i-1,j}^{t}}{2\Delta y} + v_{i,j}^{t} \frac{r_{i+1,j}^{t} - r_{i-1,j}^{t}}{2\Delta y} \right)$$

$$\overline{r}_{i,j}^{t+\Delta t} = r_{i,j}^{t} + \left(\frac{\partial r}{\partial t}\right)_{i,j}^{t} \Delta t$$

- u, v are predicted the same way

$$\overline{u}_{i,j}^{t+\Delta t} = u_{i,j}^{t} + \left(\frac{\partial u}{\partial t}\right)_{i,j}^{t} \Delta t \qquad \overline{v}_{i,j}^{t+\Delta t} = v_{i,j}^{t} + \left(\frac{\partial v}{\partial t}\right)_{i,j}^{t} \Delta t$$

MacCormack Technique

- corrector step for density

$$\left(\overline{\frac{\partial r}{\partial t}}\right)_{i,j}^{t+\Delta t} = -\left(\overline{r}_{i,j}^{t+\Delta t} \frac{\overline{u}_{i+1,j}^{t+\Delta t} - \overline{u}_{i-1,j}^{t+\Delta t}}{2\Delta x} + \overline{u}_{i,j}^{t+\Delta t} \frac{\overline{r}_{i+1,j}^{t+\Delta t} - \overline{r}_{i-1,j}^{t+\Delta t}}{2\Delta x} + \overline{r}_{i,j}^{t+\Delta t} \frac{\overline{v}_{i+1,j}^{t+\Delta t} - \overline{v}_{i-1,j}^{t+\Delta t}}{2\Delta y} + \overline{v}_{i,j}^{t+\Delta t} \frac{\overline{r}_{i+1,j}^{t+\Delta t} - \overline{r}_{i-1,j}^{t+\Delta t}}{2\Delta y} \right)$$

$$r_{i,j}^{t+\Delta t} = r_{i,j}^{t} + \left(\frac{\partial r}{\partial t}\right)_{i,j}^{av} \Delta t = r_{i,j}^{t} + \frac{1}{2}\left[\left(\frac{\partial r}{\partial t}\right)_{i,j}^{t} + \left(\overline{\frac{\partial r}{\partial t}}\right)_{i,j}^{t+\Delta t}\right]\Delta t$$

- corresponds to Heun scheme for ODEs
- other higher-order schemes, e. g. Runge-Kutta 2 or 4, could be used as well

Comments

- Lax-Wendroff and MacCormack can be used for unsteady flow
  - non-conservation form
  - conservation form
  - viscous flow
  - inviscid flow
- higher-order accuracy required to avoid numerical dissipation, artificial viscosity, numerical dispersion
- note, that viscosity is represented by second partial derivatives in the governing equations

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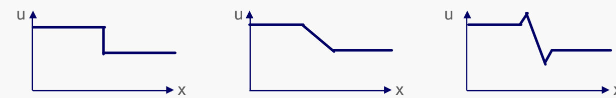
Numerical Effects

- truncation error causes dissipation and dispersion
- numerical dissipation is caused by even-order terms of the truncation error
- numerical dispersion is caused by odd-order terms
- leading term in the truncation error dominates the behavior



original function   dissipation   dispersion

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Numerical Effects

- artificial viscosity compromises the accuracy, but improves the stability of the solution
- adding artificial viscosity increases the probability of making the solution less accurate, but improves the stability
- similar to iterative solution schemes for implicit integration schemes, where a smaller number of iterations introduces “artificial viscosity”, but improves the stability

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Summary

- governing equations for compressible flow
  - continuity equation, momentum equation
  - Navier-Stokes (viscous flow) and Euler (inviscid flow)
  - discussion of conservation and non-conservation form
- explicit time-marching techniques
  - Lax-Wendroff
  - MacCormack
  - numerical aspects
- recent research in graphics

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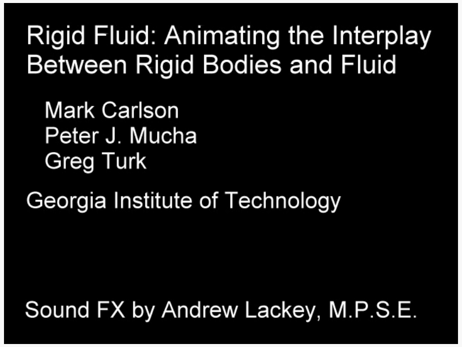
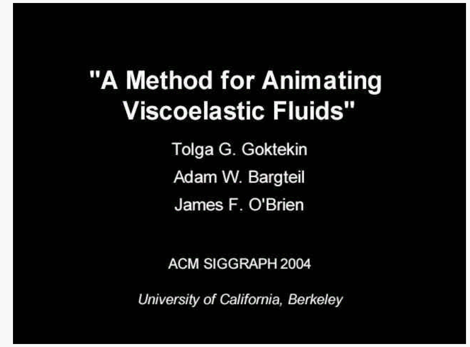
Recent Research



M. Carlson, P. J. Mucha, G. Turk, “Rigid Fluid: Animating the Interplay Between Rigid Bodies and Fluid,” Siggraph 2004.

Recent Research



T. G. Goktekin, A. W. Bargteil, J. F. O'Brien,
“A Method for Animating Viscoelastic Fluids,” Siggraph 2004.

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